Entropy and Its Variational Principle for Product Type Dynamical Systems

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Abstract

We extend the variational principle for entropies to a class of continuous maps defined on a countable product of locally compact metric spaces, which we call *product type dynamical systems*. This class includes the shifts and is closed under composition. A major consequence is the variational principle for locally compact dynamical systems without the assumption that the dynamical system is proper. We also apply our results to extend some previous formulas for the topological entropy of continuous endomorphisms of connected Lie groups without the hypothesis that the endomorphism is surjective. In the case of a linear semi-simple Lie group and of a simply-connected nilpotent Lie group, we show that the topological entropy of an endomorphism always vanishes. In particular, the entropy of a linear endomorphism of a finite dimensional vector space always vanishes. In the case of a compact Lie group, we prove that its topological entropy of an endomorphism coincides with the topological entropy of its restriction to the maximal connected and compact subgroup of the center.

1 Introduction

In this paper, we extend the variational principle for entropies to a class of continuous maps defined on a countable product of locally compact metric

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spaces, which we call *product type dynamical systems*. This class includes the shifts and is closed under composition. Thus, we generalize a result presented in [6], which states the variational principle for the shift in $\mathbb{R}^\infty$. As a major consequence of our main result, we improve the variational principle for locally compact dynamical systems presented in [8], since here we do not assume that the dynamical systems are proper. In fact, our approach is quite different from that one adopted in [8], which uses the one point compactification. Here we adapted the classical proof of variational principle due to Misiurewicz (see Theorem 8.6 of [14]) and use a different type of compactification, presented in [7]. We also apply our results to extend some previous formulas for the topological entropy of continuous endomorphisms of connected Lie groups proved in [9]. In the case of a linear semi-simple Lie group and of a simply-connected nilpotent Lie group, without assuming that the endomorphism is surjective, we prove that its topological entropy always vanishes. In particular, the entropy of a linear endomorphism of a finite dimensional vector space always vanishes. In the case of a compact Lie group, without assuming that the endomorphism is surjective, we prove that its topological entropy coincides with the topological entropy of its restriction to the maximal connected and compact subgroup of the center.

The paper is organized in the following way. In Section 2, we recall some elementary definitions related to the different types of entropies and prove some fundamental facts which are used in the next sections. In Section 3, we introduce the class of *product type dynamical systems* and prove some useful results about sequence of measures on the Borel sets of a countable product of locally compact metric space. In Section 4, we prove our main result and some applications of the topological entropy to continuous endomorphisms of some classes of Lie groups.

2 Preliminaries

This section is devoted to recall some elementary definitions related to the different types of entropies and to prove some fundamental facts which are used in the next sections.

A *dynamical system* $T : X \to X$ is a continuous map T defined over a metrizable topological space X. Recall that a family $\mathcal{A}$ of subsets of X is a *cover of* X, or simply a *cover*, when

$$X = \bigcup \mathcal{A}.$$

If the sets in $\mathcal{A}$ are disjoint, then we say that $\mathcal{A}$ is a *partition of* X. A *subcover* of $\mathcal{A}$ is a family $\mathcal{B} \subset \mathcal{A}$ which is itself a cover of X. If $Y \subset X$ and

$\mathcal{A}$ is a cover of X, then we denote by $Y \cap \mathcal{A}$ the cover of Y given by

$$Y \cap \mathcal{A} = \left\{ A \cap Y \,\middle|\, A \in \mathcal{A} \right\}.$$

We denote by $N(\mathcal{A})$ the least cardinality amongst the subcovers of $\mathcal{A}$. For $Y \subset X$, $N_Y(\mathcal{A})$ is the same as $N(Y \cap \mathcal{A})$.

Given two covers $\mathcal{A}$ and $\mathcal{B}$ of an arbitrary set X, we say that $\mathcal{A}$ is *finer then* $\mathcal{B}$ or that $\mathcal{A}$ *refines* $\mathcal{B}$ — and write $\mathcal{B} \prec \mathcal{A}$ — when every element of $\mathcal{A}$ is a subset of some element of $\mathcal{B}$. We also say that $\mathcal{B}$ is coarser then $\mathcal{A}$. The relation $\prec$ is a *preorder*, and if we identify the *symmetric* sets, we would have a *lattice*. And, as usual, $\mathcal{A} \vee \mathcal{B}$ denotes the representative of the coarsest covers of X that refines both, $\mathcal{A}$ and $\mathcal{B}$, given by

$$\mathcal{A} \vee \mathcal{B} = \left\{ A \cap B \,\middle|\, A \in \mathcal{A}, B \in \mathcal{B}, A \cap B \neq \emptyset \right\}.$$

Given a dynamical system $T : X \to X$ and a cover $\mathcal{A}$, then, for each $n \in \mathbb{N}$ we define

$$\mathcal{A}^n = \mathcal{A} \vee \cdots \vee T^{-(n-1)}(\mathcal{A}).$$

If we want to emphasise the dynamical system T, then we write $\mathcal{A}_T^n$ instead.

Consider the finite measure space $(X, \mathcal{B}, \mu)$ and a finite measurable partition $\mathcal{C}$. The *partition entropy of* $\mathcal{C}$ is

$$H_\mu(\mathcal{C}) = \sum_{A \in \mathcal{C}} \mu(A) \log \frac{1}{\mu(A)}.$$

For the dynamical system $T : X \to X$, if μ is a T-invariant finite measure, the *partition entropy of* T *with respect to* $\mathcal{C}$ is

$$h_\mu(T \mid \mathcal{C}) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\mathcal{C}^n),$$

and the *Kolmogorov-Sinai entropy of* T is

$$h_\mu(T) = \sup_{\substack{\mathcal{C}\text{: finite} \\ \text{measurable partition}}} h_\mu(T \mid \mathcal{C}).$$

Notice that our definition of the Kolmogorov-Sinai entropy does not assume that μ is a probability measure. This will prove to be useful when X is metrizable locally compact (but not necessarily compact), since, in this case, the set of probability measures is not compact in the weak-$*$ topology, while the set of finite measures with $0 \leq \mu(X) \leq 1$ is indeed compact (see the proof of Lemma 3.7).

Lemma 2.1. *Given a dynamical system $T : X \to X$ and a finite T-invariant measure μ on the Borel sets of X, then, for $\alpha \in \mathbb{R}$,*

$$h_{\alpha\mu}(T) = \alpha h_\mu(T).$$

Proof. We can assume that $\alpha \neq 0$, since $h_0(T) = 0$.

For any measurable finite partition $\mathcal{C}$,

$$\begin{aligned}\frac{1}{n}\sum_{C\in\mathcal{C}^n} \alpha\mu(C)\log\frac{1}{\alpha\mu(C)} &= \frac{\alpha}{n}\sum_{C\in\mathcal{C}^n}\mu(C)\log\frac{1}{\alpha\mu(C)} \\ &= \frac{\alpha}{n}\left(\sum_{C\in\mathcal{C}^n}\mu(C)\log\frac{1}{\mu(C)}\right) + \frac{\alpha}{n}\mu(X)\log\frac{1}{\alpha}.\end{aligned}$$

Now, we just have to take the limit for $n \to \infty$. □

Inspired on the Kolmogorov-Sinai entropy, Adler, Konheim and McAndrew introduced in [1] a purely topological concept of entropy. When X is compact, Dinaburg and Goodman proved (see [3, 5]) the *variational principle*, which states that the *topological entropy* is equal to the supremum of the Kolmogorov-Sinai entropy taken over all T-invariant probability measures. In [8], Patrão noticed that when the dynamical system admitted a one point compactification, the variational principle still holds as long as we adapt the original definition of topological entropy. In the present paper, we provide a definition of *topological entropy* which extends the previous definitions and allows us to prove the variational principle in a much broader set up.

Definition 2.2 (Cover Entropy)**.** *Given a cover $\mathcal{A}$ of a set X, the* cover entropy of $\mathcal{A}$ *is*

$$H(\mathcal{A}) = \log N(\mathcal{A}).$$

Motivated by the definition presented in [8], we shall restrict our attention to open covers of a certain type.

Definition 2.3 (Admissible Cover)**.** *In a topological space X, an open cover $\mathcal{A}$ is said to be* pre-admissible *when at least one of its elements have compact complement. An* admissible cover *is a cover of the form $\mathcal{A}^n$, where $\mathcal{A}$ is pre-admissible.*

Remark 2.4. If $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ is a compactification of $T : X \to X$, then, for any admissible cover $\mathcal{A}$ of X, there exists an open cover $\widetilde{\mathcal{A}}$ of $\widetilde{X}$, such that $\mathcal{A} = X \cap \widetilde{\mathcal{A}}$. In fact, since $\left(X \cap \widetilde{\mathcal{A}}\right)^n = X \cap \widetilde{\mathcal{A}}^n$, it is enough to assume that $\mathcal{A}$

is pre-admissible. Let $K \subset X$ be a compact set such that $A_K = X \setminus K \in \mathcal{A}$. For every $A \in \mathcal{A}$, choose an open set $\widetilde{A} \subset \widetilde{X}$ such that $A = X \cap \widetilde{A}$, taking care to choose $\widetilde{A_K} = \widetilde{X} \setminus K$. This last requirement warrants that $\widetilde{\mathcal{A}}$ covers $\widetilde{X}$.

Definition 2.5 (Topological Entropy)**.** *For a dynamical system $T : X \to X$ and a cover $\mathcal{A}$, the* topological entropy of T with respect to $\mathcal{A}$ *is*

$$h\big(T \bigm| \mathcal{A}\big) = \lim_{n\to\infty} \frac{1}{n} H\left(\mathcal{A}^n\right).$$

The topological entropy of T *is*

$$h\left(T\right) = \sup_{\mathcal{A}:\ \text{admissible}} h\big(T \bigm| \mathcal{A}\big).$$

Throughout this paper, the term *AKM entropy* refers to the original definition of entropy given by Adler, Konheim and McAndrew, while the term *topological entropy* refers to our modified definition.

Remark 2.6. Notice that as in the case of the AKM entropy and the Kolmogorov-Sinai entropy, the limit in Definition 2.5 exists thanks to the inequality

$$N\left(\mathcal{A} \vee \mathcal{B}\right) \leq N\left(\mathcal{A}\right) N\left(\mathcal{B}\right)$$

(see Theorem 4.10 in [14]).

We now state some very basic properties satisfied by the topological entropy. Most of the arguments are consequence of the following simple lemma.

Lemma 2.7. *Given a dynamical system $T : X \to X$, let $\mathcal{A}$ and $\mathcal{B}$ cover X and be such that $\mathcal{B} \prec \mathcal{A}$. Then, for all $n \in \mathbb{N}$, and every subset $Y \subset X$,*

1. *$\mathcal{B}^n \prec \mathcal{A}^n$.*
2. *$N_Y\left(\mathcal{B}^n\right) \leq N_Y\left(\mathcal{A}^n\right)$.*
3. *$h\big(T \bigm| \mathcal{B}\big) \leq h\big(T \bigm| \mathcal{A}\big)$.*

Proposition 2.8. *Consider the dynamical system $T : X \to X$, and let $k \in \mathbb{N}$. Then,*

$$h\left(T^k\right) = k h\left(T\right).$$

Proof. Let $\mathcal{A}$ be an admissible cover of X. Notice that $(\mathcal{A}^k)^n_{T^k} = \mathcal{A}^{kn}_T$. So,

$$k\frac{1}{kn}H\left(\mathcal{A}^{kn}_T\right) = \frac{1}{n}H\left((\mathcal{A}^k)^n_{T^k}\right).$$

Taking the limit for $n \to \infty$, we have that

$$kh\big(T \,\big|\, \mathcal{A}\big) = h\Big(T^k \,\Big|\, \mathcal{A}^k\Big) \leq h\left(T^k\right).$$

Now, we just have to take the supremum for all admissible covers $\mathcal{A}$ in order to conclude that

$$kh\left(T\right) \leq h\left(T^k\right).$$

On the other hand,

$$h\Big(T^k \,\Big|\, \mathcal{A}\Big) \leq h\Big(T^k \,\Big|\, \mathcal{A}^k\Big) = kh\big(T \,\big|\, \mathcal{A}\big),$$

where the first inequality follows by Lemma 2.7 and the fact that $\mathcal{A} \prec \mathcal{A}^k$. $\square$

Bowen introduced in [2] a definition of entropy which coincides with AKM's topological entropy when the dynamical system is compact metrizable. We shall not adopt here the definition Bowen gave for the non-compact case. Instead, we follow Patrão's idea of restricting our attention to metrics that arise from some compactification $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ of the original dynamical system $T : X \to X$, where X is a dense subset of the compact metrizable set $\widetilde{X}$, and $\widetilde{T}$ restricted to X is equal to T. The following result was extracted from Theorem 6.1 in [7] and shows that every separable metrizable dynamical system admits a metrizable compactification.

Proposition 2.9. *Let $T : X \to X$ be a dynamical system where X is separable and metrizable. Then, T admits a metrizable compactification $\widetilde{T} : \widetilde{X} \to \widetilde{X}$.*

Choose a metric d in X and, given $\varepsilon > 0$, denote by

$$\mathcal{B}_d\left(\varepsilon\right) = \Big\{B_d\left(\varepsilon; x\right) \,\Big|\, x \in X\Big\}$$

the family of balls of radius ε. Also, for $n \in \mathbb{N}$, we define

$$d_n(x, y) = \max_{0 \leq j < n} d(T^j x, T^j y).$$

In the literature, (n, ε)-spanning sets are usually defined as sets $E \subset X$ for which given any $x \in X$, there exists $y \in E$ such that $d_n(x, y) \leq \varepsilon$. We adopt an equivalent definition but in terms of covers, in a way that is easier to relate with our definition of topological entropy.

Definition 2.10 ((n, ε)-Spanning Set). *Let $T : X \to X$ be a dynamical system with a metric d. For a given $\varepsilon > 0$ and $n \in \mathbb{N}$, a set $E \subset X$ is a (n, ε)-*spanning set *when*

$$X = \bigcup_{x \in E} B_{d_n}(\varepsilon; x).$$

That is, the family $\left\{ B_{d_n}(\varepsilon; x) \,\middle|\, x \in E \right\}$ is a cover for X.

Definition 2.11 (d-Entropy). *Let $T : X \to X$ be a dynamical system and d a metric for X. Given $\varepsilon > 0$ and $n \in \mathbb{N}$, define*

$$h^d(T, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log N(\mathcal{B}_{d_n}(\varepsilon)),$$

and

$$h^d(T) = \sup_{\varepsilon > 0} h^d(T, \varepsilon).$$

We denote the usual Bowen entropy by $h_d(T)$.

Remark 2.12. Again, just like in the case of the topological entropy and the Kolmogorov-Sinai entropy, the limit in Definition 2.11 exists thanks to the inequality

$$N(\mathcal{B}_{d_n}(\varepsilon)) \leq N\left(\mathcal{B}_{d_q}(\varepsilon)\right) N\left(\mathcal{B}_{d_{n-q}}(\varepsilon)\right)$$

for all $q \in \mathbb{N}$ such that $0 < q < n$ (see Theorem 4.10 in [14]).

An alternative way to characterize Bowen's entropy is by using *separated sets*. Our proof of the variational principle needs this characterization.

Let $T : X \to X$ be a dynamical system with metric d. Given $\varepsilon > 0$ and $n \in \mathbb{N}$, we say that a set $S \subset X$ is (n, ε)-*separated* when for all pairs of distinct points $x, y \in S$, $d_n(x, y) > \varepsilon$. For a subset $Y \subset X$ and $\varepsilon > 0$, let us write $s(n, \varepsilon, Y)$ for the supremum amongst the cardinalities of (n, ε)-separated subsets of Y.

Lemma 2.13. *In a dynamical system $T : X \to X$, where (X, d) is a metric space, given a subset $Y \subset X$ and $\varepsilon > 0$, then, for all $n \in \mathbb{N}$,*

$$N_Y(\mathcal{B}_{d_n}(\varepsilon)) \leq s(n, \varepsilon, Y) \leq N_Y\left(\mathcal{B}_{d_n}\left(\frac{\varepsilon}{2}\right)\right).$$

Proof. The first inequality follows by the following claim and by the existence (through Zorn's lema) of maximal (n, ε)-separated sets.

Claim. *If $E \subset Y$ is a maximal (n, ε)-separated set, then*

$$\mathcal{B}_E = \left\{ Y \cap B_{d_n}(\varepsilon; x) \,\middle|\, x \in E \right\}$$

is a cover of Y.

If $\mathcal{B}_E$ is not a cover, then, taking $y \in Y \setminus \bigcup_{x \in E} B_{d_n}(\varepsilon; x)$, we have that the set $E \cup \{y\}$ is (n, ε)-separated, infringing the maximality of the set E.

For the second inequality, if $S \subset Y$ is a (n, ε)-separated set, and $E \subset Y$ is a $(n, \varepsilon/2)$-spanning set for Y, then for each $s \in S$, there exists an $e(s) \in E$ such that $d_n(s, e(s)) \leq \frac{\varepsilon}{2}$, since the balls centered in points of E with radius $\varepsilon/2$ do cover Y. This mapping is injective, since if $e(s_1) = e(s_2)$, then $d_n(s_1, s_2) \leq \varepsilon$. In this case, since S is (n, ε)-separated, we must have $s_1 = s_2$. □

Proposition 2.14. *For a dynamical system $T : X \to X$, where (X, d) is a metric space,*

$$h^d(T) = \sup_{\varepsilon > 0} \lim_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, X).$$

Proof. It is immediate by Lemma 2.13. Just take the log, divide by n and take the limit in

$$N(\mathcal{B}_{d_n}(\varepsilon)) \leq s(n, \varepsilon, X) \leq N\left(\mathcal{B}_{d_n}\left(\frac{\varepsilon}{2}\right)\right).$$

□

For the definition of d-entropy, we use the families $\mathcal{B}_{d_n}(\varepsilon)$. Notice that those families are not the same as $[\mathcal{B}_d(\varepsilon)]^n$. The following lemma shows that the families $[\mathcal{B}_d(\varepsilon)]^n$ would work as well, making the d-entropy and the topological entropy much easier to compare.

Lemma 2.15. *Let $T : X \to X$ be a dynamical system with metric d. Then,*

$$h^d(T) = \sup_{\varepsilon > 0} h\left(T \,\middle|\, \mathcal{B}_d(\varepsilon)\right).$$

Proof. It is enough to show that

$$[\mathcal{B}_d(\varepsilon)]^n \prec \mathcal{B}_{d_n}(\varepsilon) \prec \left[\mathcal{B}_d\left(\frac{\varepsilon}{2}\right)\right]^n.$$

Indeed, this would imply that

$$\frac{1}{n} \log N([\mathcal{B}_d(\varepsilon)]^n) \leq \frac{1}{n} \log N(\mathcal{B}_{d_n}(\varepsilon)) \leq \frac{1}{n} \log N\left(\left[\mathcal{B}_d\left(\frac{\varepsilon}{2}\right)\right]^n\right).$$

Now, we just need to make $n \to \infty$ and take the supremum for $\varepsilon > 0$.

It is immediate that

$$\mathcal{B}_{d_n}(\varepsilon) \subset [\mathcal{B}_d(\varepsilon)]^n,$$

since every ball in the metric d_n has the form

$$B_d(\varepsilon; x) \cap \cdots \cap T^{-(n-1)} B_d\left(\varepsilon; T^{n-1}x\right).$$

On the other hand, a set $A \in \left[\mathcal{B}_d\left(\frac{\varepsilon}{2}\right)\right]^n$ has the form

$$A = B_d\left(\frac{\varepsilon}{2}; x_0\right) \cap \cdots \cap T^{-(n-1)} B_d\left(\frac{\varepsilon}{2}; x_{n-1}\right).$$

In particular, if $A \neq \emptyset$, $d(T^j x_0, x_j) < \frac{\varepsilon}{2}$. So,

$$B_d\left(\frac{\varepsilon}{2}; x_j\right) \subset B_d\left(\varepsilon; T^j x_0\right).$$

Therefore,

$$A \subset B_d(\varepsilon; x_0) \cap \cdots \cap T^{-(n-1)} B_d\left(\varepsilon; T^{n-1}x_0\right) \in \mathcal{B}_{d_n}(\varepsilon).$$

□

The following lemma states the existence of the *Lebesgue number* in a form which is easy to apply to the construction of covers' refinements. Essentially, every cover for a compact set can be refined by some cover given by the balls of a given fixed radius.

Lemma 2.16 (Lebesgue Number)**.** *Suppose (X, d) be a metric space, which admits a compactification $\left(\widetilde{X}, \widetilde{d}\right)$. Let $\mathcal{A} = X \cap \widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}}$ is an open cover of $\widetilde{X}$. Then, there exists $\varepsilon > 0$ such that*

$$\mathcal{A} \prec \mathcal{B}_d(\varepsilon).$$

Proof. Let

$$C_\varepsilon = \left\{x \in \widetilde{X} \;\middle|\; \exists A \in \widetilde{\mathcal{A}},\, B_{\widetilde{d}}(\varepsilon; x) \subset A\right\}$$

be the set of all $x \in \widetilde{X}$ such that the balls centered in x with radius ε is contained in some element of $\widetilde{\mathcal{A}}$. Now, we will find $\varepsilon > 0$ such that $\widetilde{X} = C_\varepsilon$.

Claim. $\widetilde{X} = \bigcup_{\varepsilon > 0} C_\varepsilon$.

For each $x \in \widetilde{X}$, there is a $A \in \widetilde{\mathcal{A}}$ containing x. This means that there exists $\varepsilon > 0$ with $B_{\widetilde{d}}(\varepsilon; x) \subset A$.

Claim. $C_\varepsilon \subset \operatorname{int}\left(C_{\frac{\varepsilon}{2}}\right)$.

Just notice that if $x \in C_\varepsilon$, then

$$B_{\widetilde{d}}\left(\frac{\varepsilon}{2}; x\right) \subset C_{\frac{\varepsilon}{2}}.$$

In fact, for $y \in B_{\widetilde{d}}\left(\frac{\varepsilon}{2}; x\right)$, we have that $B_{\widetilde{d}}\left(\frac{\varepsilon}{2}; y\right) \subset B_{\widetilde{d}}(\varepsilon; x)$. This last ball is contained in some element of $\widetilde{\mathcal{A}}$, since $x \in C_\varepsilon$.

Joining both claims, we have that

$$\widetilde{X} = \bigcup_{\varepsilon > 0} \operatorname{int}\left(C_\varepsilon\right).$$

Since $\widetilde{X}$ is compact, there exists $\varepsilon > 0$ such that $\widetilde{X} = C_\varepsilon$. This means that $\widetilde{\mathcal{A}} \prec \mathcal{B}_{\widetilde{d}}(\varepsilon)$. Taking the intersection with X,

$$\mathcal{A} = X \cap \widetilde{\mathcal{A}} \prec X \cap \mathcal{B}_{\widetilde{d}}(\varepsilon).$$

Now, we just have to observe that since

$$B_d\left(\varepsilon; x\right) = X \cap B_{\widetilde{d}}(\varepsilon; x)$$

for any $x \in X$, then

$$\mathcal{B}_d\left(\varepsilon\right) \subset X \cap \mathcal{B}_{\widetilde{d}}(\varepsilon).$$

Therefore,

$$\mathcal{A} \prec \mathcal{B}_d\left(\varepsilon\right).$$

□

Finally, we relate the topological entropy and the d-entropy.

Proposition 2.17. *Let $T : X \to X$ be a dynamical system admitting a compactification $\widetilde{T} : \widetilde{X} \to \widetilde{X}$. If d is the restriction to X of a metric $\widetilde{d}$ in $\widetilde{X}$, then*

$$h\left(T\right) \leq h^d\left(T\right) = \sup_{\widetilde{\mathcal{A}}\text{: open}} h\left(T \,\middle|\, X \cap \widetilde{\mathcal{A}}\right),$$

where the supremum is taken over all open covers of $\widetilde{X}$.

Proof. The family $\mathcal{D} = \left\{B_{\widetilde{d}}(\varepsilon; x) \,\middle|\, x \in X\right\}$ covers $\widetilde{X}$, and is such that $\mathcal{B}_d\left(\varepsilon\right) = X \cap \mathcal{D}$. This implies that

$$h^d\left(T\right) \leq \sup_{\widetilde{\mathcal{A}}\text{: open}} h\left(T \,\middle|\, X \cap \widetilde{\mathcal{A}}\right).$$

In a similar fashion, Remark 2.4 implies that

$$h(T) \leq \sup_{\widetilde{\mathcal{A}}:\text{ open}} h\Big(T \,\Big|\, X \cap \widetilde{\mathcal{A}}\Big).$$

Now, Lemma 2.16 implies that for any open cover of $\widetilde{X}$, $\widetilde{\mathcal{A}}$, there exists $\varepsilon > 0$ such that

$$X \cap \widetilde{\mathcal{A}} \prec \mathcal{B}_d(\varepsilon).$$

And this means that

$$h\Big(T \,\Big|\, X \cap \widetilde{\mathcal{A}}\Big) \leq h\Big(T \,\Big|\, \mathcal{B}_d(\varepsilon)\Big).$$

Taking the supremum in ε and applying the Lemma 2.15, we conclude that

$$h\Big(T \,\Big|\, X \cap \widetilde{\mathcal{A}}\Big) \leq h^d(T).$$

Taking the supremum for all open covers $\widetilde{\mathcal{A}}$,

$$\sup_{\widetilde{\mathcal{A}}:\text{ open}} h\Big(T \,\Big|\, X \cap \widetilde{\mathcal{A}}\Big) \leq h^d(T).$$

□

Remark 2.18. When the families $\mathcal{B}_d(\varepsilon)$ are admissible covers, Lemma 2.15 warrants that

$$h^d(T) \leq h(T).$$

Nonetheless, even when d is the restriction of a metric given in some compactification of the system, we have that $\mathcal{B}_d(\varepsilon)$ is not always admissible. As an example, just take $\mathcal{B}_d\left(\frac{1}{2}\right)$ in $(0,1)$ with the Euclidean metric. No set in $\mathcal{B}_d\left(\frac{1}{2}\right)$ will ever have compact complement and therefore is not pre-admissible. Depending on the transformation T, $\mathcal{B}_d\left(\frac{1}{2}\right)$ might not be admissible either.

When the dynamical system is compact, we know that the d-entropy does not depend on the metric d. The following corollary to Proposition 2.17 extends this result.

Corollary 2.19. *Let $T : X \to X$ be a dynamical system admitting a compactification $\widetilde{T} : \widetilde{X} \to \widetilde{X}$. If d and c are the restriction to X of metrics $\widetilde{d}$ and $\widetilde{c}$ in $\widetilde{X}$, then*

$$h^d(T) = h^c(T).$$

3 Product Type Dynamical System

In this paper, we are interested in the following generalization of locally compact dynamical systems, which we call *product type* dynamical system. First we introduce some notations.

Consider $X = \prod_{i=1}^{\infty} X_j$, where X_j are topological spaces, which will be denoted just by $X = \prod X_j$. We denote by X^n the space $X_1 \times \cdots \times X_n$ and by π_n the canonical projection

$$\begin{array}{rccc} \pi_n : & X & \to & X^n \\ & (x_j) & \mapsto & (x_1, \dots, x_n) \end{array} .$$

A set $A \subset X$ is said to be π_n-measurable when there is a Borel set $B \subset X^n$ such that $A = \pi_n^{-1}(B)$. We say that an application $f : X \to Y$ is π_n-measurable when for any measurable $A \subset Y$ the set $f^{-1}(A)$ is π_n-measurable.

Definition 3.1 (Product Type Dynamical System)**.** *Let $T : X \to X$ be a dynamical system with $X = \prod X_j$. We say that $T : X \to X$ is a* product type dynamical system *when*

1. *Each X_j is locally compact.*
2. *For every $n \in \mathbb{N}$ there is a $m = m(n) \in \mathbb{N}$, with $m \geq n$, such that $\pi_n \circ T$ is π_m-measurable.*
3. *The dynamical system $T : X \to X$ admits a compactification $\widetilde{T} : \widetilde{X} \to \widetilde{X}$, with $\widetilde{X} = \prod \widetilde{X}_j$, where each $\widetilde{X}_j$ is compact and metrizable and X_j is a dense subset of $\widetilde{X}_j$.*

We note that the countable product of locally compact systems $T_j : X_j \to X_j$, denoted by $\prod T_j$, is a product type dynamical system. When the X_j are all the same, we can consider the shift, which is in fact a product type dynamical system. Notice that the class of product type dynamical systems is closed under composition.

Example 3.2. Consider the set of real formal series $\left\{ \sum_{j=0}^{\infty} a_j x^j \,\middle|\, a_j \in \mathbb{R} \right\}$, which can be canonically identified to $\mathbb{R}^{\infty}$. The formal derivation given by

$$\frac{d}{dx} \sum_{j=0}^{\infty} a_j x^j = \sum_{j=0}^{\infty} (j+1) a_{j+1} x^j$$

is a product type dynamical system. In fact, it is the composition of the shift by the product $\prod T_j$, where

$$T_j a_j x^j = j a_j x^j.$$

The higher formal derivatives are also product type dynamical systems, since they are compositions of the first formal derivation.

Let us show some properties that a product type dynamical system satisfies. First we shall mention some elementary properties, like their metrizability and some topological properties satisfied by the projections π_n. Then we shall state some non-elementary properties.

Proposition 2.9 shows that a locally compact dynamical system admits a compactification. In particular, a locally compact dynamical system $T : X \to X$ is isomorphic to a *product type* dynamical system. To see that, we just have to identify X with $X \times \{0\}^{\mathbb{N}}$. The following proposition, whose proof will be omitted, gives us a metric for the compactification that shall be useful in the sequel.

Proposition 3.3. *Let $T : X \to X$ be a product type dynamical system. Then, there exists a metric $\widetilde{d}_j$ for each $\widetilde{X}_j$ such that $\widetilde{X}_j$ has diameter lower then 1, and*

$$\widetilde{d}((x_j),(y_j)) = \sup_{j=1}^{\infty} \frac{1}{j} \widetilde{d}_j(x_j, y_j)$$

is compatible with the product topology in $\widetilde{X} = \prod \widetilde{X}_j$.

Let us call the metric $\widetilde{d}$ from Proposition 3.3 (or its restriction to X) a *product type metric* for the product type dynamical system.

Lemma 3.4. *Let $B \subset X^n$ be a Borel set. Then,*

$$\partial\left(\pi_n^{-1}(B)\right) = \pi_n^{-1}(\partial B).$$

Proof.

Claim.

$$\overline{\pi_n^{-1}(B)} = \pi_n^{-1}(\overline{B}).$$

Since π_n is continuous,

$$\overline{\pi_n^{-1}(B)} \subset \pi_n^{-1}(\overline{B}).$$

On the other hand, if $(x_j) \in \pi_n^{-1}(\overline{B})$, then there is a $(b_1^k, \dots, b_n^k) \in B$ converging to $(x_1, \dots, x_n)$. Letting $b_j^k = x_j$ for $j > n$, we have that $(b_j^k) \in \pi_n^{-1}(B)$ converges to x. This implies that $x \in \overline{\pi_n^{-1}(B)}$. That is,

$$\pi_n^{-1}(\overline{B}) \subset \overline{\pi_n^{-1}(B)}.$$

Using the above assertion for B and for B^c, it follows that

$$\begin{aligned} \partial\left(\pi_n^{-1}(B)\right) &= \overline{\pi_n^{-1}(B)} \cap \overline{\pi_n^{-1}(B^c)} \\ &= \pi_n^{-1}(\overline{B}) \cap \pi_n^{-1}(\overline{B^c}) \\ &= \pi_n^{-1}(\partial B). \end{aligned}$$

□

Proposition 3.5. *Let $T : X \to X$ be a product type dynamical system, then X is a Borel set inside $\widetilde{X}$. In particular, all measures on the Borel sets of X are Radon measures.*

Proof.

Claim. *Each X_j is a Borel set inside $\widetilde{X}_j$.*

Since X_j is a locally compact second countable Hausdorff space, there is a countable base for the topology made of sets with compact closure. In particular, X_j can be written as a countable union of compact sets. In its turn, compact sets of X_j are in fact compact sets of $\widetilde{X}_j$. As an arbitrary union of compact sets, X_j is a Borel set of $\widetilde{X}_j$.

Using the previous claim, the sets $\widetilde{\pi}_n^{-1}(X_1 \times \cdots \times X_n)$ are Borel sets of $\widetilde{X}$. Since,

$$\widetilde{\pi}_n^{-1}(X_1 \times \cdots \times X_n) \downarrow X,$$

it turns out that X is also a Borel set of $\widetilde{X}$.

The topology in X is given by sets of the form $A \cap X$, with $A \subset \widetilde{X}$ open. So, the Borel sets of X will be sets of the form $B \cap X$, where $B \subset \widetilde{X}$ is a Borel set of $\widetilde{X}$. So, every measure μ defined on the Borel sets of X can be extended to a measure on the Borel sets of $\widetilde{X}$, given by

$$\widetilde{\mu}(B) = \mu(B \cap X).$$

Since $\widetilde{\mu}$ is a Radon measure and the Borel sets of X are also Borel sets of $\widetilde{X}$, we have that all of them can be approximated "from inside" by compact sets of $\widetilde{X}$. Now, we just need to notice that the compact sets of $\widetilde{X}$ contained in X are in fact compact sets of X. □

The main difficulty when studying non-locally compact dynamical system is the fact that the Riesz representation Theorem cannot be used. In the remaining of this section, we state and prove assertions that will be used to show the *variational principle* for the product type dynamical systems, pretty much the same way the Riesz representation Theorem and its consequences are used in the classical demonstration of Misiurewicz (see Theorem 8.6 of [14]).

Lemma 3.6. *The π_n-measurable functions $f : X \to \mathbb{R}$ are in bijection with the measurable functions $g : X^n \to \mathbb{R}$ through the correspondence*

$$g \mapsto g \circ \pi_n.$$

Proof. Since evidently $g \circ \pi_n$ is π_n-measurable, the non-trivial part is the proof that any π_n-measurable $f : X \to \mathbb{R}$ has the form $g \circ \pi_n$ for some measurable g.

The application g is well-defined by the relation $g(\pi_n(x)) = f(x)$. In fact, if $\pi_n(x) = \pi_n(y)$, then, letting $\alpha = f(x)$, since f is π_n-measurable, there exists a Borel set $B \subset X^n$, such that

$$f^{-1}(\alpha) = \pi_n^{-1}(B).$$

This implies that $\pi_n(y) = \pi_n(x) \in B$. That is, $f(y) = \alpha$. Also notice that $g : X^n \to \mathbb{R}$ is measurable. In fact, since f is π_n-measurable, given a Borel set $A \subset \mathbb{R}$, there is a Borel set $B \subset X^n$, such that

$$\pi_n^{-1}\left(g^{-1}(A)\right) = f^{-1}(A) = \pi_n^{-1}(B).$$

Since π_n is surjective, this implies that $g^{-1}(A) = B$. $\square$

Lemma 3.7. *Let $T : X \to X$ be a product type dynamical system, and σ_k a sequence of probabilities on the Borel sets of X. Then, letting*

$$\mu_k = \frac{1}{k} \sum_{j=0}^{k-1} \sigma_k \circ T^{-j},$$

there is a measure μ on the Borel sets of X, such that

1. $0 \leq \mu(X) \leq 1$.
2. *The measure μ is T-invariant.*
3. *For $n \in \mathbb{N}$, if $C \subset X$ is π_n-measurable and satisfies $\mu(\partial C) = 0$, then*

$$\liminf_{k \to \infty} \mu_k(C) \leq \mu(C).$$

Proof. Notice that each X^n is a locally compact metrizable Hausdorff space. The *Riesz representation Theorem* (Theorem 2.14 in [13]) states that we can identify the linear functionals on the real functions in X^n vanishing at infinity with the finite measures on the Borel sets of X^n. This, together with *Alaoglu's Theorem* (Theorem 2.5.2 in [11]) and the fact that the positive linear functionals with operator norm lower than or equal to 1 are a closed set in the weak-* topology, imply that the set of positive measures with total measure lower than or equal to 1 is a compact set in the weak-* topology.

We first construct the measure μ, and then show that it meets the stated conditions. Define the sequence $s(0,j) = j$. If we have $s(n-1,j)$ defined, we can choose $s(n,j)$, a sub-sequence of $s(n-1,j)$, and a positive measure μ^n with $0 \leq \mu^n(X^n) \leq 1$ such that

$$\mu_{s(n,j)} \circ \pi_n^{-1} \to \mu^n$$

in the weak-* topology. Now, using the Kolmogorov consistency Theorem (Theorem A.6 in [4]), there is a measure μ defined on the Borel sets of X, such that

$$\mu \circ \pi_n^{-1} = \mu^n.$$

Now, we show that μ has the required properties.

- Item (1).

  Notice that $0 \leq \mu^1(X_1) \leq 1$. But, $\mu(X) = \mu(\pi_1^{-1}(X_1)) = \mu^1(X_1)$.

- Item (2).

Claim. *For all continuous limited $f : X^n \to \mathbb{R}$,*

$$\int f \, \mathrm{d}(\mu \circ \pi_n^{-1}) = \int f \, \mathrm{d}(\mu \circ T^{-1} \circ \pi_n^{-1}).$$

That is,

$$\mu \circ \pi_n^{-1} = \mu \circ T^{-1} \circ \pi_n^{-1}.$$

Let $m = m(n)$ (definition 3.1). Notice that $m \geq n$. Since f is limited, it is easy to check using μ_k's definition, that

$$\int f \, \mathrm{d}(\mu_{s(m,j)} \circ T^{-1} \circ \pi_n^{-1}) \to \int f \, \mathrm{d}\mu^n = \int f \, \mathrm{d}(\mu \circ \pi_n^{-1}).$$

On the other hand, we have that $f \circ \pi_n \circ T$ is π_m-measurable. Using Lemma 3.6, $f \circ \pi_n \circ T$ can be written as $g \circ \pi_m$ for some measurable application $g : X^m \to \mathbb{R}$. So,

$$\begin{aligned}\int f \,\mathrm{d}(\mu_{s(m,j)} \circ T^{-1} \circ \pi_n^{-1}) &= \int f \circ \pi_n \circ T \,\mathrm{d}\mu_{s(m,j)} \\ &= \int g \circ \pi_m \,\mathrm{d}\mu_{s(m,j)} \\ &= \int g \,\mathrm{d}(\mu_{s(m,j)} \circ \pi_m^{-1}) \\ &\to \int g \,\mathrm{d}(\mu \circ \pi_m^{-1}) \\ &= \int g \circ \pi_m \,\mathrm{d}\mu \\ &= \int f \circ \pi_n \circ T \,\mathrm{d}\mu \\ &= \int f \,\mathrm{d}(\mu \circ T^{-1} \circ \pi_n^{-1}).\end{aligned}$$

That is, for all n, and all continuous limited $f : X^n \to \mathbb{R}$,

$$\int f \,\mathrm{d}(\mu \circ \pi_n^{-1}) = \int f \,\mathrm{d}(\mu \circ T^{-1} \circ \pi_n^{-1}).$$

So,

$$\mu \circ \pi_n^{-1} = \mu \circ T^{-1} \circ \pi_n^{-1}.$$

The previous claim together with the uniqueness provided by the Kolmogorov consistency Theorem implies that $\mu = \mu \circ T^{-1}$.

■ Item (3).

There is a $B \subset X^n$ such that $C = \pi_n^{-1}(B)$. Notice that, by Lemma 3.4, $\partial C = \pi_n^{-1}(\partial B)$. So, by Remark (3) in page 149 of [14], since $\mu \circ \pi_n^{-1}(\partial B) = 0$, we have that

$$\mu_{s(n,j)}(C) = \mu_{s(n,j)} \circ \pi_n^{-1}(B) \to \mu \circ \pi_n^{-1}(B) = \mu(C).$$

In particular,

$$\liminf_{k \to \infty} \mu_k(C) \leq \mu(C).$$

□

Lemma 3.8. *Let $T : X \to X$ be a product type dynamical system, μ a finite T-invariant Radon measure in X and $\varepsilon > 0$. Then, there is $n \in \mathbb{N}$, and a finite measurable partition $\mathcal{C}$, made of π_n-measurable sets such that for each $C \in \mathcal{C}$,*

$$\mu(\partial C) = 0.$$

And also such that the diameter of each $C \in \mathcal{C}$ on a product type metric is less then ε. Furthermore, for all $k \in \mathbb{N}$, each member of $\mathcal{C}^k$ is $\pi_{m^k(n)}$-measurable and has null measure border, where $m^k(n)$ is the function m applied to n k times.

Proof. Choose $n > \frac{1}{\varepsilon}$. Let $\widetilde{c}$ be the metric

$$\widetilde{c}((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max_{j=1}^{n} \widetilde{d}(x_j, y_j).$$

Let c be the restriction of $\widetilde{c}$ to X^n. For each $x \in X^n$, there is ε_x with $\frac{\varepsilon}{2} \leq \varepsilon_x \leq \varepsilon$, such that $B_c(\varepsilon_x; x) \subset X^n$ has border with measure $\mu \circ \pi_n^{-1}$ equals to 0. In fact, for different radius $\varepsilon' > 0$, the borders of $B_c(\varepsilon'; x)$ are disjoint. The family of borders of balls centered in x and radius between $\frac{\varepsilon}{2}$ and ε is an uncountable family of measurable disjoint sets. Therefore, there is an infinity of them with measure $\mu \circ \pi_n^{-1}$ equals to 0.

The balls $B_{\widetilde{c}}\left(\frac{\varepsilon}{2}; x\right)$ with $x \in X^n$, cover $\widetilde{X}_1 \times \cdots \times \widetilde{X}_n$, and therefore they have a finite sub-cover. Using the fact that

$$B_c\left(\frac{\varepsilon}{2}; x\right) = X \cap B_{\widetilde{c}}\left(\frac{\varepsilon}{2}; x\right),$$

we find a cover for X^n of the form

$$B_c\left(\frac{\varepsilon}{2}; x_1\right), \dots, B_c\left(\frac{\varepsilon}{2}; x_k\right).$$

In particular, the balls $B_c(\varepsilon_{x_1}; x_1), \dots, B_c(\varepsilon_{x_k}; x_k)$ do cover X. We can then assume that this cover is minimal. Denote

$$A_j = \pi_n^{-1}(B_c(\varepsilon_{x_1}; x_1)).$$

Using Lemma 3.4, we have that the borders of A_j have measure μ equals to 0.

Now, let's construct $\mathcal{C}$. Let

$$C_j = A_j \setminus (A_1 \cup \cdots \cup A_{j-1}).$$

The family $\mathcal{C} = \{C_1, \dots, C_k\}$ is a partition where each of its elements have diameter less then ε in any product type metric, because we have choosen $n > \frac{1}{\varepsilon}$. Each C_j is such that

$${C_j}^c = {A_j}^c \cup A_1 \cup \cdots \cup A_{j-1}.$$

The border of a set is just the border of its complement. To conclude that the border of each C_j has measure μ equals to 0, we just have to show that the border of finite unions is inside the unions of each individual border.

Claim. *For a family* $Y_1, \ldots, Y_s \subset X^n$,

$$\partial (Y_1 \cup \cdots \cup Y_s) \subset \partial Y_1 \cup \cdots \cup \partial Y_s.$$

Since

$$\overline{Y_1 \cup \cdots \cup Y_s} = \overline{Y_1} \cup \cdots \cup \overline{Y_s}$$

and

$$\operatorname{int}(Y_1 \cup \cdots \cup Y_s) \supset \operatorname{int}(Y_1) \cup \cdots \cup \operatorname{int}(Y_s),$$

it follows that

$$\begin{aligned}\partial (Y_1 \cup \cdots \cup Y_s) &= \overline{Y_1} \cup \cdots \cup \overline{Y_s} \setminus (\operatorname{int}(Y_1 \cup \cdots \cup Y_s)) \\ &\subset \overline{Y_1} \cup \cdots \cup \overline{Y_s} \setminus (\operatorname{int}(Y_1) \cup \cdots \cup \operatorname{int}(Y_s)) \\ &\subset \left(\overline{Y_1} \setminus \operatorname{int}(Y_1)\right) \cup \cdots \cup \left(\overline{Y_s} \setminus \operatorname{int}(Y_s)\right) \\ &= \partial Y_1 \cup \cdots \cup \partial Y_s.\end{aligned}$$

Using the previous claim, not only the border of each element in $\mathcal{C}$ is contained in a finite union of sets with measure μ equals to 0, but it also follows that the same is true for the border of any element in $\mathcal{C}^k$, since T preserves μ. By the definition of product type dynamical system, it is immediate that the elements of $\mathcal{C}^k$ are $\pi_{m^k(n)}$-measurable. □

4 Variational Principle

The proof for the following theorem is inspired on Misiurewicz proof of the variational principle, presented for example in Theorem 8.6 of [14].

Theorem 4.1. *Let* $T : X \to X$ *be a product type dynamical system. Then,*

$$\sup_\mu h_\mu(T) = h(T) = \min_d h_d(T).$$

The infimum is attained when d is any metric that can be extended to $\widetilde{X}$.

Proof. The proof that

$$\sup_\mu h_\mu(T) \leq h(T)$$

is identical to classical Misiurewicz's proof. One just has to notice that the open covering constructed there is in fact admissible, and also that as in the classical case, we can use the formula given by Proposition 2.8.

Therefore, using Proposition 2.17 and the above inequality,

$$\sup_{\mu} h_{\mu}(T) \leq h(T) \leq h^{d}(T),$$

where d is the restriction of some distance on $\widetilde{X}$. We claim that it is sufficient to show that

$$h^{d}(T) \leq \sup_{\mu} h_{\mu}(T).$$

In fact, this implies that

$$\sup_{\mu} h_{\mu}(T) = h(T) = h^{d}(T),$$

where d is the restriction of some distance on $\widetilde{X}$. Now, observe that since X is a Borel subset of $\widetilde{X}$ (see Proposition 3.5), the supremum of the Kolmogorov-Sinai entropies taken over the ergodic measures coincides with the supremum taken over the invariant measures (see the proof of Theorem 1 of [12], page 311). Hence using Proposition 1.4 of [6] and the fact that, for any metric d,

$$h_{d}(T) \leq h^{d}(T),$$

we conclude that

$$\sup_{\mu} h_{\mu}(T) \leq \min_{d} h_{d}(T) \leq \min_{d} h^{d}(T) \leq h^{d}(T) = \sup_{\mu} h_{\mu}(T).$$

In order to show that $h^{d}(T) \leq \sup_{\mu} h_{\mu}(T)$, Corollary 2.19 allows us to assume that d is the metric in X given by Proposition 3.3.

Using Proposition, 2.14, it remains to show that, for each fixed $\varepsilon > 0$ and each sequence of (n, ε)-separated sets E_n, there is a Radon measure μ, which is T-invariant and has total measure lower then or equal to 1, and there is a finite measurable partition $\mathcal{C}$, such that

$$\log \# E_n \leq H_{\mu}(\mathcal{C}^{n}).$$

Let us first build up the measure μ, and show that it satisfies the desired conditions. Define

$$\sigma_n = \frac{1}{\# E_n} \sum_{x \in E_n} \delta_x,$$

where δ_x is the Dirac measure with support in x. Also define

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} \sigma_n \circ T^{-j}.$$

Now, take the T-invariant Radon measure μ from Lemma 3.7, and then choose $\mathcal{C}$ as in Lemma 3.8.

Claim. $\log \#E_n \leq H_{\sigma_n}(\mathcal{C}^n)$.

Let $C \in \mathcal{C}^n$. If $x, y \in C$, then, there exist $C_0, \ldots, C_{n-1} \in \mathcal{C}$ such that $T^j x, T^j y \in C_j$ for $j = 0, \ldots, n-1$. Since each element of $\mathcal{C}$ has diameter less then ε, we have that $d_n(x, y) < \varepsilon$. So, C can contain at most one element of E_n. That is, $\sigma_n(C) = 0$ or $\sigma_n(C) = \frac{1}{\#E_n}$. Therefore,

$$H_{\sigma_n}(\mathcal{C}^n) = \log \#E_n.$$

To pass from σ_n to μ_n, we just proceed exactly as in Misiurewicz's classical proof. We estimate $H_{\mu_n}(\mathcal{C}^q)$ in terms of $H_{\sigma_n}(\mathcal{C}^q)$.

Just like in the classical proof, it follows that

$$\frac{q}{n}\log \#E_n \leq H_{\mu_n}(\mathcal{C}^q) + \frac{2q^2}{n}\log k.$$

Since each element of $\mathcal{C}^q$ is π_m-measurable for some m, and has border with measure μ equals to 0, then, by Lemma 3.7, item (3),

$$\begin{aligned}\limsup_{n\to\infty} \frac{q}{n}\log \#E_n &\leq \liminf_{n\to\infty} H_{\mu_n}(\mathcal{C}^q) \\ &\leq H_{\mu}(\mathcal{C}^q).\end{aligned}$$

Now, we just have to divide it by q and take the limit to get to

$$\limsup_{n\to\infty} \frac{1}{n}\log \#E_n \leq h_\mu\big(T \,\big|\, \mathcal{C}\big).$$

$\square$

The following result, which is an immediate consequence of the previous theorem, improves the variational principle for locally compact dynamical systems presented in [8], since it does not assume that the dynamical system is proper.

Corollary 4.2. *Let $T : X \to X$ be a locally compact metrizable dynamical system. Then,*

$$\sup_{\mu} h_{\mu}(T) = h(T) = \min_{d} h_d(T).$$

The infimum is attained when d is any metric that can be extended to $\widetilde{X}$.

In conjunction with the Variational Principle, Proposition 2.17 gives an alternative characterization of the topological entropy.

Corollary 4.3. *Let $T : X \to X$ be a product type dynamical system. Then,*

$$h(T) = \sup_{\widetilde{\mathcal{A}}:\ open} h\Big(T \,\Big|\, X \cap \widetilde{\mathcal{A}}\Big),$$

where the supremum is taken over all open covers of $\widetilde{X}$.

It is well known, in the compact case, that the entropy of the product of two dynamical systems is the sum of their entropies. The variational principle allow us to extend this result.

Proposition 4.4. *Consider the product system $T = \prod T_j$, where each $T_j : X_j \to X_j$ is a locally compact metrizable dynamical system. Then*

$$h\left(\prod T_j\right) = \sum h(T_j)$$

Proof. Let $d(x, y) = \sup \frac{1}{j} d_j(x_j, y_j)$ be a product type metric in $\prod X_j$ (see Proposition 3.3). Then, for each $\varepsilon > 0$,

$$\mathcal{B}_d(\varepsilon) = \mathcal{B}_{d_1}(\varepsilon) \times \mathcal{B}_{d_2}(2\varepsilon) \times \mathcal{B}_{d_3}(3\varepsilon) \times \cdots .$$

And notice that for $n > \frac{1}{\varepsilon}$, $\mathcal{B}_{d_n}(n\varepsilon) = \{X_n\}$. More generally, since T acts on each coordinate, for every $k \in \mathbb{N}$,

$$[\mathcal{B}_d(\varepsilon)]^k = [\mathcal{B}_{d_1}(\varepsilon)]^k \times [\mathcal{B}_{d_2}(2\varepsilon)]^k \times [\mathcal{B}_{d_3}(3\varepsilon)]^k \times \cdots .$$

Again, for $n > \frac{1}{\varepsilon}$, $[\mathcal{B}_{d_n}(n\varepsilon)]^k = \{X_n\}$. It follows that

$$N\left([\mathcal{B}_d(\varepsilon)]^k\right) = N\left([N(\mathcal{B}_{d_1}(\varepsilon))]^k\right) N\left([\mathcal{B}_{d_2}(2\varepsilon)]^k\right) N\left([\mathcal{B}_{d_3}(3\varepsilon)]^k\right) \cdots .$$

Therefore, for $n > \frac{1}{\varepsilon}$,

$$\begin{aligned} h\Big(T \,\Big|\, \mathcal{B}_d(\varepsilon)\Big) &= \sum_{j=1}^{n} h\Big(T_j \,\Big|\, \mathcal{B}_{d_j}(j\varepsilon)\Big) \\ &= \sum h\Big(T_j \,\Big|\, \mathcal{B}_{d_j}(j\varepsilon)\Big). \end{aligned}$$

Now, the variational principle along with Lemma 2.15 imply that

$$\begin{aligned} h\,(T) &= \sup_{\varepsilon>0} h\Big(T \,\Big|\, \mathcal{B}_d\,(\varepsilon)\Big) \\ &= \lim_{\varepsilon\downarrow 0} h\Big(T \,\Big|\, \mathcal{B}_d\,(\varepsilon)\Big) \\ &= \lim_{\varepsilon\downarrow 0} \sum h\Big(T_j \,\Big|\, \mathcal{B}_{d_j}\,(j\varepsilon)\Big) \\ &= \sum \lim_{\varepsilon\downarrow 0} h\Big(T_j \,\Big|\, \mathcal{B}_{d_j}\,(j\varepsilon)\Big) \\ &= \sum h\,(T_j), \end{aligned}$$

where the forth equation follows by the Monotone Convergence Theorem. □

We finish considering the entropy of continuous endomorphisms of Lie groups. We extend some results of [9] to endomorphisms which are not necessarily surjective. For a given Lie group G, its toral component $T(G)$ is the maximal connected and compact subgroup of the center of G, which is isomorphic to a torus.

Lemma 4.5. *Let G be a connected Lie group and $\phi : G \to G$ be a continuous endomorphism. There exists a natural number n such that ϕ restricted to $H = \phi^n(G)$ is surjective. If H is closed, then*

$$h\,(\phi) = h\,(\phi|_H).$$

Proof. For the first claim, we first consider the induced Lie algebra endomorphism $\phi' : \mathfrak{g} \to \mathfrak{g}$ given by the differential of ϕ at the identity of G. Since ϕ' is a linear map, there is a natural number n such that $(\phi')^n\mathfrak{g} = (\phi')^{n+1}\mathfrak{g}$. Putting $H = \phi^n(G)$, we have that its Lie algebra is given by $\mathfrak{h} = (\phi')^n\mathfrak{g}$. We have that $\phi(H) = H$, since both are connected subgroups and the Lie algebra of $\phi(H)$ is $\phi'\mathfrak{h} = \mathfrak{h}$. Thus ϕ restricted to H is surjective.

For the second claim, we first observe that H contains the recurrent set $\mathcal{R}(\phi)$. If H is closed, it contains the closure $\overline{\mathcal{R}(\phi)}$. Thus, using the Variational Principle, we get that

$$h\,(\phi) = h\left(\phi|_{\overline{\mathcal{R}(\phi)}}\right) = h\,(\phi|_H).$$

□

In the following two results, we use that the image $\varphi(G)$ of a semi-simple Lie group G by a continuous endomorphism φ is a semi-simple subgroup. Indeed, we have that $\varphi(G)$ is isomorphic to $G/\ker\varphi$ and $\ker\varphi$ is a normal subgroup.

Proposition 4.6. *Let G be a connected linear semi-simple Lie group and $\phi : G \to G$ be a continuous endomorphism. Then*

$$h(\phi) = 0.$$

Proof. Consider $H = \phi^n(G)$ given by Lemma 4.5. Denoting $\varphi = \phi^n$, we have that $H = \varphi(G)$ is a connected linear semi-simple Lie group. Thus H is closed (see Proposition 3.5 of [10]). Using Lemma 4.5, and applying Theorem 5.4 of [9] for ϕ restricted to H, we get that

$$h(\phi) = h(\phi|_H) = h\left(\phi|_{T(H)}\right) = 0.$$

Now the proposition follows, since $T(G) = 1$. $\square$

Proposition 4.7. *Let G be a connected and compact Lie group and consider $\phi : G \to G$ a continuous endomorphism. Then*

$$h(\phi) = h\left(\phi|_{T(G)}\right).$$

Proof. Consider $H = \phi^n(G)$ given by Lemma 4.5. Denoting $\varphi = \phi^n$, we have that $H = \varphi(G)$ is a connected and compact Lie group. Using Lemma 4.5, and applying Proposition 6.7 of [9] for ϕ restricted to H, we get that

$$h(\phi) = h(\phi|_H) = h\left(\phi|_{T(H)}\right).$$

If we prove that $T(H) \subset T(G)$, we have that

$$h(\phi) = h\left(\phi|_{T(H)}\right) \leq h\left(\phi|_{T(G)}\right) \leq h(\phi).$$

In order to show that $T(H) \subset T(G)$, we consider the canonical endomorphism $\overline{\varphi} : G/T(G) \to G/T(G)$ induced by φ, whose image is given by

$$\frac{HT(G)}{T(G)} \simeq \frac{H}{T(G) \cap H}.$$

Since $G/T(G)$ is a semi-simple Lie group, we have that the above image is a semi-simple Lie group, showing that $T(G) \cap H = T(H)$ and completing the proof. $\square$

Proposition 4.8. *Let G be a simply-connected nilpotent Lie group and consider $\phi : G \to G$ a continuous endomorphism. Then*

$$h(\phi) = 0.$$

In particular, the entropy of a linear endomorphism of a finite dimensional vector space always vanishes.

Proof. Consider $H = \phi^n(G)$ given by Lemma 4.5. Denoting $\varphi = \phi^n$, we have that $H = \varphi(G)$ is a simply-connected nilpotent Lie group. Therefore H is a closed subgroup. Using Lemma 4.5, and applying Theorem 4.3 of [9] for ϕ restricted to H, we get that

$$h(\phi) = h(\phi|_H) = h\left(\phi|_{T(H)}\right) = 0.$$

Now the proposition follows, since $T(G) = 1$. □

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